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1986 J. Phys. A: Math. Gen. 19 L1197

(http://iopscience.iop.org/0305-4470/19/18/015)

J. Phys. A: Math. Gen. 19 (1986) L1197-L1199. Printed in Great Britain

LETTER TO THE EDITOR

Rigid clusters enumeration

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Received 9 October 1986

Abstract. Animal counting techniques suggest that the fraction of rigid animals among all animals with s sites on a triangular lattice varies roughly as $s^{1/2}$ $(0.46 \pm 0.01)^s$.

Recently the mechanical behaviour of tenuous structures (as FI gels or corroded sheets) has received much attention. De Gennes (1976) originally suggested a percolation analysis using the formal analogy between the elastic problem and conductivity. Despite the fact that, due to the different tensorial characters of the two problems, the critical properties of elastic moduli near percolation threshold are different from those of conductivity, the de Gennes' analysis has been the starting point of numerous works on this subject. Most of them simulate the tenuous material using a bond percolation network.

Up to now, two important limit models have been studied.

(*a*) *The bond-bending model.* Here potential energy and elastic moduli near the threshold are controlled by the angles between two connected bonds (angular elasticity). In this model the elastic threshold is the same as for the conductivity but the critical exponent of elastic moduli, T, is different from the exponent t of the conductivity. The scaling hypothesis $T = t + 2\nu$ (Roux (1986) with earlier literature) recently received confirmation from computer simulations by Zabolitzky *et al* (1986).

(*b*) *The central force model.* Here the elastic energy depends on the distance between connected sites, each bond acting as a spring. In this model the threshold and critical exponents (Lemieux *et al* 1985) are definitely different from those of the bond-bending problem. Note that, for the central force problem, honeycomb and square lattices do not exhibit rigidity. A recent paper (Day *et al* 1986) emphasises the peculiar geometrical properties of that model and especially considers a new class of clusters, the rigid clusters, which are very different from the percolation ones. In this letter we thought it interesting to enumerate the rigid clusters and to compare their number with that of classical percolation animals.

In this letter, a cluster of bonds is rigid if every site linked to it is constrained in all directions in the linear approximation of the central force model.

There are two kinds of clusters of bonds: saturated (all the bonds between two nearest-neighbour sites exist) and unsaturated clusters. A given cluster may be rigid only if the corresponding saturated cluster is rigid. A cluster of s sites needs b' constraints to be rigid. In two dimensions, b' satisfies the relation $2s - b' = 3$. The number, b, of bonds in a rigid cluster is greater than or equal to b' so that it must satisfy

$$2s - b \leqslant 3. \qquad (1)$$

0305-4470/86/181197+03\$02.50

However, for $s \geqslant 10$, there are clusters which are rigid only in a non-linear analysis and do not satisfy (1). We choose to restrict ourselves to clusters rigid in the linear approximation. So, using the standard algorithm (Redner 1982) for site cluster generation (a saturated cluster is, in fact, a site cluster), we generate all saturated clusters up to a size $s_{\max}$ and for each of them we check the rigidity. In order to improve the speed of our algorithm this test is performed in three steps.

(i) In the first, one rejects the cluster if there is a single-connected site;

(ii) then one keeps only the clusters for which (1) is satisfied;

(iii) after the two preceding steps, most of non-rigid clusters are rejected (cf table 1). The rigidity of the remaining clusters is then checked using an algorithm which deals with superbonds (any rigid cluster between two sites) and triangles of super-bonds.

After this third step, all the rigid saturated clusters have been kept. Then, from each of them and using the same algorithm, we build up all unsaturated and rigid clusters. From (1), it is easily seen that one has obtained, in that way, all rigid clusters up to $(2s_{\max}-2)$ bonds. Nevertheless, some non-rigid (in the linear approximation) clusters are not rejected. Their relative number, compared with the total number of rigid clusters, is less than 1% from $s=10$ to $s=13$.

Tables 1 and 2 summarise our numerical results for rigid clusters up to 13 sites for saturated ones and up to 24 bonds for all of them. In table 1, we give the number n_s of rigid saturated clusters and, for comparison, the number a_s of site percolation animals and the number n_s^* of clusters kept after the two first steps of our algorithm. Before any numerical analysis of these data, one may remark that the number of saturated clusters is much lower than that of percolation animals. This is due to the very severe geometrical constraints which must be satisfied to get a rigid cluster.

Table 1. Rigid saturated clusters. Number of percolation animals, a_s, and rigid saturated animals, n_s, as a function of number s of sites. n_s^* gives the number of clusters satisfying the first two steps of our algorithm.

		Number of clusters	
Sites	a_s	n_s^*	n_s
1	1	0	0
2	3	3	3
3	11	2	2
4	44	3	3
5	186	6	6
6	814	14	14
7	3 652	31	31
8	16 689	69	69
9	77 359	151	151
10	362 671	341	335
11	1 716 033	795	747
12	8 182 213	1905	1671
13	39 267 086	4667	3749

If we assume that the ratio A_s between the number of rigid saturated clusters and that of percolation animals of the same size s is of the form:

$$A_s = A_0 s^{\theta} \lambda^{s} \tag{2}$$

Table 2. Rigid clusters of bonds. Number of rigid clusters as a function of number of bonds. Note the oscillation between even and odd terms.

Bonds b	Number n_b	Bonds b	Number n_b
1	3	13	135
2	0	14	6
3	2	15	460
4	0	16	27
5	3	17	1 798
6	0	18	143
7	6	19	7 235
8	0	20	687
9	14	21	30 587
10	0	22	3 327
11	42	23	136 159
12	1	24	16 589

a standard data analysis (ratio plot, etc) gives

$$\lambda = 0.46 \pm 0.01 \qquad \theta \sim 0.5.$$

In table 2 we give the number n_b of rigid clusters (saturated and unsaturated) of b bonds. These data slightly overestimate (within 1%) the exact number of animals rigid in the linear approximation. The number of rigid clusters oscillates with the parity of b. The amplitude of this oscillation decreases for larger and larger clusters and is expected to be negligible for animals of sufficient size. We assume that n_b asymptotically obeys the law:

$$n_b = \text{constant} \times b^{\theta_1} \lambda_1 b.$$

We have computed λ_1 separately for even and odd values of b. In both cases we find

$$\lambda_1 = 2.2 \pm 0.1.$$

Due to the oscillation of the data and to the uncertainty ($\sim 1\%$) in the exact number of rigid unsaturated clusters, it is not possible to determine θ in that case.

In conclusion, we made an enumeration of the rigid clusters on a triangular lattice, the number of which is considerably lower than that of percolation animals on the same lattice. The next step, which uses that enumeration in standard series expansions, is now in progress.

This letter would not have been written without the friendly attention of D Stauffer. We are grateful to him for constructive discussions and critical reading of the manuscript.

References

Day A R, Tremblay R R and Tremblay A-M S 1986 *Phys. Rev. Lett.* **56** 2501
de Gennes P G 1976 *J. Physique Lett.* **37** L1
Lemieux M A, Breton P and Tremblay A-M S 1985 *J. Physique Lett.* **46** L1
Redner S 1982 *J. Stat. Phys.* **29** 309
Roux S 1986 *J. Phys. A: Math. Gen.* **19** L351
Zabolitzky J G, Bergman D J and Stauffer D 1986 *J. Stat. Phys.* **44** 211